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THE DOLLAR VALUE OF IMPROVED RELIABILITY

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FOREWORD

This is the third of three monographs in which ARINC Research Corporation has presented results of studies concerning the interrelationships of equipment reliability, maintenance, and cost. The first of this series, Monograph 7, described a method for determining a preventive maintenance schedule based upon part replacement prior to in-service failure. This method assumed a lack of any measure of equipment deterioration. Monograph 8 proposed a method of scheduling preventive maintenance designed to minimize average hourly maintenance cost. A basic assumption in this instance was the availability of some measure of equipment deterioration.

The development in the following pages is concerned particularly with the savings to be realized from scheduling of preventive maintenance and from improvements in reliability. Two primary requirements are first considered: (1) a criterion for determining a preventive maintenance schedule, and (2) the associated curves relating average cost per mean life to the fundamental cost parameter. On this basis, the study develops a nomograph for use in computing the maximum cost increment that can be paid for improved reliability.

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1. INTRODUCTION

For approximately two years, ARINC Research Corporation has been engaged in studies designed to relate reliability, maintenance, and cost. This is the third monograph in the series resulting from these studies.

The first of these papers, ARINC Research Monograph No. 7, "Relationship Between Equipment Reliability, Preventive Maintenance Policy, and Operating Costs," described a method for determining a preventive maintenance schedule based upon part replacement prior to in-service failure. The model used in Monograph 7 related equipment operating time between maintenance actions under such a schedule to the expected average hourly cost of maintenance. This model was based on the assumption that a measure of equipment deterioration was not available. Monograph 8, entitled "A Model for Scheduling Maintenance Utilizing Measures of Equipment Performance," described a method for scheduling preventive maintenance to minimize expected average hourly maintenance cost when some measure of deterioration is available. In both of these monographs, the emphasis was on scheduling, and relatively minor attention was given to the cost element.

The present monograph focuses attention on the savings to be realized from scheduling of preventive maintenance and from

improvements in reliability. The scheduling system of
Monograph 7 is employed in the discussion and illustration of
the theory, since the mathematical formulation in Monograph 7
is simpler than that required for the model described in Mono-
graph 8. However, as will be noted in the conclusions, the
major portion of the theoretical development in this paper is
also valid for other preventive maintenance systems.

2. DEFINITION OF THE PROBLEM

As was done in Monographs 7 and 8, the discussion in this monograph will be phrased in terms of a single-part equipment, though generalization is obviously possible.

As a basis for the statement of the problem, let it be assumed that a standard part has been in use in a particular application under an optimum preventive maintenance schedule determined by some stated criterion. For example, the model in Monograph 7 called for repair of in-service failures as they occur and replacement of the part after h hours of equipment operation, where h is selected to yield the minimum expected average hourly cost.

Suppose, further, that an improved part becomes available -- with the word "improved" having reference to the part reliability, and "reliability" in turn meaning longer mean life or less variability about this mean, or both. This monograph proposes to answer the questions:

(1) How much more can the user afford to pay for the improved reliability?

(2) Is the answer to Question (1) sensitive to the exact nature of the part failure pattern, or is it sensitive only to certain characteristics of the time-to-failure density function?

In the development of answers to the foregoing questions, the following assumptions will be made:

(1) Each repair of an in-service failure and each preventive maintenance action consists of a part replacement.

(2) Standard and improved parts are interchangeable. Hence the switch to the improved part adds to the cost only the increase in the purchase price of the part, since there is no change in the amount of labor involved in a repair action.

(3) The criterion for scheduling preventive maintenance depends only on the failure pattern of the part, the cost of a preventive maintenance action, and the cost of the repair of an in-service failure. The assumption that these two cost parameters are constant is believed to be a relatively minor restriction for purposes of maintenance scheduling and estimation of expected average cost.

3. METHOD OF SOLUTION

The questions posed in this monograph can be answered by developing -- in the manner of Monograph 7 -- a number of relationships between cost parameters and time-to-failure density parameters, together with certain additional relationships which take account of reliability and cost changes resulting from replacement of a standard part by an improved part.

Most of the required answers can be obtained by considering two fundamental relationships. The first is the formula for the cost increment which would yield a maintenance cost for the improved part exactly equal, on the average, to that of the standard part. This, then, gives the maximum which could be paid for improved reliability without increasing over-all costs; or, in other words, the "break-even" point.

The second relationship is the expected average hourly cost expressed as a function of all pertinent cost and time-to-failure density parameters. This relationship is needed in development of the formula for the cost increment at the break-even point, as well as in determination of the expected average hourly cost in the event an improved part can be purchased for some lesser cost increase. It is obvious that the costs involved in this discussion are all influenced by the

maintenance policy. As noted previously, the present treatment
uses illustrations phrased in terms of the optimum maintenance,
in the manner of Monograph 7; however, this approach does not
cause a loss of generality except to the extent indicated in
the assumptions listed in Section 2.

3.1 Development of the Formulas

It is convenient initially to express the formulas for
the two relationships in general form, without specification
of the exact density function. Later, the results will be
discussed for the normal, gamma, and exponential densities.
This initial treatment will refer to the mean life, the stand-
ard deviation, and the coefficient of variation, in addition
to the cost parameters and the formulas developed in Mono-
graph 7.

3.1.1 Notation

The following notation will be used, with time measured
in hours and cost in dollars. (For convenient reference,
this notation is reproduced on the foldout page at the end of
the monograph.)

Item	Standard Part	Improved Part
Mean time to failure	m	mv
Standard deviation	σ	σu
Coefficient of variation	ζ_1	ζ_2
Time between preventive maintenance actions	h_1	h_2
Cost parameters:		
Repair of an in-service failure	k_1	c_1
Cost of a preventive maintenance part replacement	k_2	c_2
Cost ratio	$k = \dfrac{k_2}{k_1}$	$c = \dfrac{c_2}{c_1}$
Improved-part cost increment		δ
Expected average cost per mean life (optimum maintenance)	$k_1 \omega$	$c_1 y$

3.1.2 General Cost Parameter Relationships

Since it was assumed that the mechanics of repair are identical for the standard and the improved parts, the only changes in cost parameters will result from the improved-part cost increment, δ. This gives the following cost parameter relationships:

$$c_1 = k_1 + \delta, \qquad c_2 = k_2 + \delta, \text{ and}$$

$$c = \frac{c_2}{c_1} = \frac{k_2 + \delta}{k_1 + \delta} = \frac{k + \delta/k_1}{1 + \delta/k_i} = \frac{k + x}{1 + x}$$

where $x = \delta/k_1$, the improved-part cost increment expressed as
a fraction of the cost of the repair of an in-service failure
for a standard part. It is clear that the cost increment, δ,
cannot be computed directly in terms of dollars in the general
case. Instead, the formula for x will be developed and δ will
then be computed in any particular application of the method.

At the break-even point, the average hourly cost for the
improved part must be equal to that for the standard part.
This condition is expressed by the equation

$$\frac{c_1 y}{mv} = \frac{k_1 \omega}{m}$$

which reduces to

$$c_1 y = k_1 v \omega.$$

In this form, it is apparent that the relationship is depend-
ent on the ratio of mean lives, v, and not on mean life values
m and mv themselves.

This equation can be modified as follows:

$$\frac{(k_1 + \delta)y}{k_1} = v\omega$$

$$(1 + x)y = v\omega.$$

Now x can be expressed in terms of c and k.

$$c = \frac{k + x}{1 + x}$$

$$c + cx = k + x$$

$$x = \frac{c - k}{1 - c} .$$

Then $\left(1 + \dfrac{c - k}{1 - c}\right) y = v\omega$, or

$$(1 - k)y = v\omega (1 - c), \text{ which gives}$$

$$c = 1 - \frac{1 - k}{v\omega} y.$$

Thus, the condition that the average hourly cost be the same for the improved part as for the standard part implies that the cost ratio for the improved part, c, is a linear function of y, the average cost per mean life of the improved part when the unit of cost is taken to be the cost of the repair of an in-service failure of the improved part.

In addition to the above break-even requirement which yielded the c,y linear relationship, it is necessary to require that c and y be chosen in such a way that they represent an optimum preventive maintenance policy. For illustration in the present discussion, the optimum is taken in the sense of Monograph 7.

It is well to note here that c and y are related to each other in the same way as are k and ω, since they refer to the same values for the improved and standard parts, respectively. As indicated in Monograph 7, the algebraic relationship between c and y is very complex for the two densities considered -- normal and gamma -- and series solutions were required. However, the c, y curves were not drawn in that monograph since the emphasis was on time between preventive maintenance actions rather than on cost ratio vs. average cost relationships. Therefore, these curves will be developed in the present monograph. To emphasize the generality of the method, it is well to repeat that these c,y (or k, ω) curves could be based on some criterion other than that of Monograph 7.

The desired solution will then be obtained as the intersection of the appropriate c,y curve and the line,

$$c = 1 - \frac{1-k}{v\,\omega}\,y.$$

This line is conveniently drawn by connecting the points

$$\left(c = 1,\ y = 0\right) \text{ and } \left(c = 0,\ y = \frac{v\omega}{1-k}\right).$$

Details of the mechanics of the process will be given later.

3.1.3 The Choice Between Type I and Type II Maintenance

Monograph 7 defined Type I maintenance as repair of in-service failures as they occur, but without any preventive-maintenance part replacement prior to failure. Type II

maintenance, by definition, also provided for repair of in-service failures as they occur, but, in addition, it included part replacement at fixed intervals of equipment operating time, prior to the failure of the part, as a preventive maintenance procedure. The optimum interval between preventive maintenance actions was determined on the condition that Type II maintenance was to be used. As a result, in the Monograph 7 development there are some cost-parameter density combinations for which the optimum interval is computed and shown on the graphs but where Type II maintenance is actually more expensive than Type I maintenance. It is obvious that where Type II maintenance is more costly than Type I, the latter should be used, and this policy is followed in the present monograph.

Whether Type II maintenance is more costly than Type I maintenance can be determined merely by observing when ω or y exceeds unity. Since ω is expressed in units of k_1, if $\omega = 1$ it means that the average cost of the Type II maintenance is equal to the cost of the repair of one in-service failure per mean life; if $\omega > 1$, then Type II maintenance costs more; and if $\omega < 1$, the implication is that Type II maintenance costs less. Since the expected average cost of Type I maintenance per mean life is precisely the cost of the repair of one in-service failure, if $\omega \leq 1$, Type II maintenance is cheaper and is used; while, if $\omega > 1$, Type I maintenance is cheaper and it is used. Identical statements can be made for y as for ω,

since it is the same kind of cost for the improved part as ω
is for the standard part.

The algebraic implication of this discussion is summarized
by saying that all curves should stop at ω and $y = 1$; and that,
in preference to consideration of values greater than 1, the
type of maintenance should be changed from Type II to Type I --
a step which would change ω or y to unity. With respect to
the linear relationship which expresses the break-even condi-
tion, this limitation on the range of ω and y implies the fol-
lowing conditions:

$\underline{\omega}$	$\underline{y}$	Linear Equation
<1	<1	$c = 1 - \dfrac{1-k}{v\omega} y$
<1	$=1$	$c = 1 - \dfrac{1-k}{v\omega}$
$=1$	<1	$c = 1 - \dfrac{1-k}{v} y$
$=1$	$=1$	$c = 1 - \dfrac{1-k}{v}$.

These four equations will be used later in the discussions of
four different cases which can arise. These cases are distin-
guished by the cost and density parameter relationships and
the type of maintenance.

3.1.4 The Change in the Density Function with an Increase in Reliability

Improved reliability depends on, or can result from, suitable changes in any one or more of the parameters which determine the density function. For example, in the case of normal density, the parameters in the equation are the mean and the variance. For this density, an increase in the mean or a decrease in the variance, or both, could result in reduced maintenance cost. Increasing the mean reduces the frequency of failure and, hence, the cost; while decreasing the variance improves the accuracy of predicting time of failure and thus enhances the potential for saving through Type II preventive maintenance, i.e., part replacement prior to failure. The situation is similar for the gamma density, but the parameters are not usually expressed in such convenient terms as the mean and variance. In this monograph, the dollar value of improved reliability will be computed for a few cases for the normal density and the gamma density, and the discussion will include the general method required to compute the dollar value for any simultaneous changes in the two density parameters.

3.2 Development of the Formulas for Specific Density Functions

3.2.1 Parameter Relationships for the Normal Density

The usual representation of the normal density function is

$$u(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\xi)^2}{2\sigma^2}} , \quad -\infty < t < \infty$$

where ξ is the mean and σ^2 is the variance, σ being the standard deviation. In Monograph 7, it was convenient to assume $\xi = 1$, but for purposes of the present treatment it is necessary to drop this assumption. As a result of this change, certain simple adjustments in units occur when cost figures are used. The explanation is that formulas developed in Monograph 7 expressed costs per mean life, and the monetary unit was the cost of repair of an in-service failure. In the present treatment, clock time must replace the mean-life unit, and the monetary unit must be similarly modified to permit comparisons between densities associated with parts which differ in density parameters and cost. This will be apparent in the following development of the mathematical relationships involved.

In the present case, by use of the notation given previously, the two density functions are derived as follows:

$$\text{Standard part} \quad \frac{1}{\sigma \sqrt{2\pi}} \, e^{-\frac{(t - m)^2}{2\sigma^2}} \quad , \text{ and}$$

$$\text{Improved part} \quad \frac{1}{\sigma u \sqrt{2\pi}} \, e^{-\frac{(t - mv)^2}{2u^2\sigma^2}} \quad .$$

The coefficients of variation are

$$\zeta_1 = \frac{\sigma}{m} \, , \quad \zeta_2 = \frac{u\sigma}{vm} = \frac{u}{v} \, \zeta_1 \, .$$

The c,y or k,ω curves are identified by the coefficient of variation for the normal density. Hence, the k,ω pair must correspond to some point on the ζ_1 curve and the c,y pair must correspond to some point on the ζ_2 curve; in addition, either pair must serve as the coordinates of a point on the break-even line (the equation for which was derived in Section 3.1.2, page 8).

3.2.2 Parameter Relationships for the Gamma Density

The general form of the gamma function is

$$u(t) = \frac{1}{\alpha! \beta^{\alpha+1}} t^\alpha e^{-t/\beta} \qquad 0 \leq t < \infty$$

with $\beta > 0$ and $\alpha + 1 > 0$. Since changing β merely changes the scales on the axes, it was possible in Monograph 7 to let $\beta = 1$ without any loss of generality. This assumption, which is similar to the assumption that $\xi = 1$ for the normal distribution, cannot be made in the present treatment. Instead, it is necessary to relate both parameters for the standard and improved versions. For this purpose, the notation will be selected to parallel that used in the case of the normal distribution. However, since the mean and variance are not the parameters in the gamma density function, as they were in the normal, it is necessary to write the formulas for these two moments for u(t) as given above. Thus, for the gamma density,

$$\xi = \beta(\alpha + 1)$$

$$\sigma = \beta \sqrt{\alpha + 1},$$

using ξ and σ to denote the mean and the standard deviation, respectively, as before.

Let α_1, β_1, ξ_1, and σ_1 relate to the standard part, and α_2, β_2, ξ_2, and σ_2 relate to the improved part, as before. Also, let u and v have similar meanings: $\sigma_2 = \upsilon\sigma_1$ and $\xi_2 = v\xi_1$. It is convenient to let $\gamma_i = \alpha_i + 1$, $i = 1,2$ to match the notation of Monograph 7. Then

$$u_i(t) = \frac{1}{(\gamma_1-1)! \, \beta_1^{\gamma_1}} \, t^{\gamma_1-1} \, e^{-t/\beta_1} \qquad 0 \le t < \infty,$$

and β_i, $\gamma_i > 0$, $i = 1,2$. In this form, the mean, standard deviation, and coefficient of variation are, respectively,

mean: $\quad \xi_1 = \beta_1 \gamma_1$

standard deviation: $\quad \sigma_1 = \beta_1 \sqrt{\gamma_1}$, and

coefficient of variation: $\quad \zeta_1 = \dfrac{1}{\sqrt{\gamma_1}}$.

For the standard and improved versions, the parameters are

$$\xi_1 = m = \beta_1 \gamma_1 \qquad\qquad \xi_2 = mv = \beta_2 \gamma_2$$

$$\sigma_1 = \sigma = \beta_1 \sqrt{\gamma_1} \qquad\qquad \sigma_2 = u\sigma = \beta_2 \sqrt{\gamma_2}$$

$$\zeta_1 = \frac{\sigma}{m} = \frac{1}{\sqrt{\gamma_1}} \qquad\qquad \zeta_2 = \frac{u\sigma}{vm} = \frac{1}{\sqrt{\gamma_2}}$$

Then the values of β_2 and γ_2 can be expressed in terms of β_1, γ_1, u, and v by the following algebra:

$$\xi_2 = vm = v\beta_1\gamma_1 = \beta_2\gamma_2$$

$$\sigma_2 = u\sigma_1 = u\beta_1\sqrt{\gamma_1} = \beta_2\sqrt{\gamma_2}$$

$$\frac{v}{u}\sqrt{\gamma_1} = \sqrt{\gamma_2}$$

$$\frac{v^2}{u^2}\gamma_1 = \gamma_2$$

$$\beta_2 = \frac{v\beta_1\gamma_1}{\gamma_2} = \frac{v\beta_1\gamma_1 u^2}{v^2\gamma_1} = \frac{u^2\beta_1}{v}$$

$$\zeta_2 = \frac{\sigma_2}{\xi_2} = \frac{u\sigma_1}{v\xi_1} = \frac{u}{v}\zeta_1 .$$

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Thus, given β_1, γ_1, u, and v, it is necessary to carry out the computations

$$\xi_1 = \beta_1 \gamma_1 , \quad \xi_2 = v\xi_1 = v\beta_1 \gamma_1$$

$$\zeta_1 = \frac{1}{\sqrt{\gamma_1}}, \quad \zeta_2 = \frac{u}{v}\zeta_1 = \frac{u}{v\sqrt{\gamma_1}}$$

$$\beta_2 = \frac{u^2}{v}\beta_1 , \quad \gamma_2 = \frac{v^2}{u^2}\gamma_1 .$$

The parameters with subscript 1 determine the curves for k,ω and those with subscript 2 determine the curves for c,y.

4. AVERAGE COST PER MEAN LIFE

4.1 The Average Cost Under Optimum Preventive Maintenance Policy as in Monograph 7

It was shown in Monograph 7 (page 25) that the average cost per unit of time for Type II maintenance is

$$\frac{da(h)}{dh}$$

under optimum selection of h where a(h) is the average number of in-service failures in the time interval of length h between preventive maintenance actions. In this form, cost is expressed in units of the cost of the repair of one in-service failure. Hence, the average cost in dollars per unit of time is

$$k_1 \frac{da(h)}{dh} \; .$$

Now the unit of time was taken to be one mean life for the normal density. However, for the gamma distribution, there were γ units of time per mean life. Hence, the cost in dollars per mean life is given by

$$k_1 \frac{da(h)}{dh} \quad \text{for the normal density, and}$$

$$k_1 \gamma \frac{da(h)}{dh} \quad \text{for the gamma density.}$$

4.2 The Average Cost Under Optimum Preventive Maintenance for the Normal Distribution

For the normal distribution, the expression for $\frac{da(h)}{dh}$ is

$$\frac{da(h)}{dh} = \frac{1}{\sigma\sqrt{2\pi}} \sum_{j=1}^{\infty} \frac{1}{\sqrt{j}}\, e^{-\frac{(h-j\xi)^2}{2j\sigma^2}}$$

$$= \sum_{j=1}^{\infty} \frac{1}{\sigma\sqrt{j}}\, n\left(\frac{h-j\xi}{\sigma\sqrt{j}}\right)$$

where $n\left(\frac{h-j}{\sigma\sqrt{j}}\right)$ is the value of the ordinate of the normal density of mean zero and unit variance at the abscissa $\frac{h-j\xi}{\sigma\sqrt{j}}$. This expression was computed as one step in determining the h,k curves in Monograph 7.

4.3 The Average Cost Under Optimum Preventive Maintenance for the Gamma Distribution

The formula for $\frac{da(h)}{dh}$ for the gamma distribution with γ a positive integer is given on page 67 of Monograph 7 as

$$\frac{da(h)}{dh} = \sum_{n=0}^{\infty} n\, e^{-h} \left[-\sum_{j=0}^{\gamma-1} \frac{h^{n\gamma+j}}{(n\gamma+j)!} + \sum_{j=0}^{\gamma-1} \frac{h^{n\gamma+j-1}}{(n\gamma+j-1)!} \right] .$$

Since this was not needed separately in the computation of the h,k curves in Monograph 7, the expression was not simplified for easy evaluation. For present purposes, it is useful to do this.

The two inner summations can be combined as follows:

$$- \sum_{j=0}^{\gamma-1} \frac{h^{n\gamma+j}}{(n\gamma+j)!} + \sum_{j=0}^{\gamma-1} \frac{h^{n\gamma+j-1}}{(n\gamma+j-1)!}$$

$$= - \frac{h^{n\gamma+\gamma-1}}{(n\gamma+\gamma-1)!} - \sum_{j=0}^{\gamma-2} \frac{h^{n\gamma+j}}{(n\gamma+j)!} + \frac{h^{n\gamma-1}}{(n\gamma-1)!} + \sum_{j=1}^{\gamma-1} \frac{h^{n\gamma+j-1}}{(n\gamma+j-1)!}$$

$$= \frac{h^{n\gamma-1}}{(n\gamma-1)!} - \frac{h^{n\gamma+\gamma-1}}{(n\gamma+\gamma-1)!} ,$$

the two summation terms being equal but opposite in sign.

Substituting this for the equivalent two summations in the expression for $\frac{da(h)}{dh}$ gives

$$\frac{da(h)}{dh} = \sum_{n=0}^{\infty} n \, e^{-h} \frac{h^{n\gamma-1}}{(n\gamma-1)!} - \frac{h^{n\gamma+\gamma-1}}{(n\gamma+\gamma-1)!}$$

$$= \sum_{n=0}^{\infty} n \, e^{-h} \frac{h^{n\gamma-1}}{(n\gamma-1)!} - \sum_{n=0}^{\infty} n \, e^{-h} \frac{h^{n\gamma+\gamma-1}}{(n\gamma+\gamma-1)!}$$

In the second summation, let $n = j-1$ and note that the first term of the first summation vanishes, so the index can start at $n = 1$. These changes give

$$\frac{da(h)}{dh} = \sum_{n=1}^{\infty} n \, e^{-h} \frac{h^{n\gamma-1}}{(n\gamma-1)!} - \sum_{j=1}^{\infty} (j-1) \, e^{-h} \frac{h^{j\gamma-1}}{(j\gamma-1)!}$$

$$= \sum_{j=1}^{\infty} e^{-h} \frac{h^{j\gamma-1}}{(j\gamma-1)!} \, .$$

Since the time unit is this case was selected so that γ is one mean life, the average cost per mean life in units of k_1 is

$$\gamma \sum_{j=1}^{\infty} e^{-h} \frac{h^{j\gamma-1}}{(j\gamma-1)!}$$

It is noted that the summation consists of terms from a Poisson distribution. This holds when the parameter γ is a positive integer.

4.4 Relationship Between Average Cost per Unit Time and the Cost Ratio

The expressions used to compute average cost per unit time under Type II maintenance give costs as functions of h, the time between scheduled, preventive-maintenance, part replacements. However, it is necessary for the present study to relate such average costs to the cost ratio -- the ratio of

the cost of one such preventive maintenance action to the cost of the repair of an in-service failure. That is, it would be advantageous to express ω as a function of k for the standard part and to express y as a function of c for the improved part. The mathematical complexity of the expressions precludes this. Instead it is easier to use a parametric representation in which ω and k (or y and c) are each expressed as functions of h, dependent of course on the parameters of the underlying time-to-failure density function. Associated values of ω and k (or y and c) are obtained by substituting a selected value of h in the two formulas for ω and k (or y and c). The formulas for average cost per unit time were derived above and those for the cost ratio are given in Monograph 7. These formulas are set forth below in terms of ω and k for the normal and gamma densities.

Normal

$$\omega = \sum_{j=1}^{\infty} \frac{1}{\sigma\sqrt{j}} n\left(\frac{h - j\xi}{\sigma\sqrt{j}}\right)$$

$$k = \sum_{j=1}^{\infty} \left[\frac{h}{\sigma\sqrt{j}} n\left(\frac{h - j\xi}{\sigma\sqrt{j}}\right) - N\left(\frac{h - j\xi}{\sigma\sqrt{j}}\right)\right]$$

Gamma

$$\omega = \gamma \sum_{j=1}^{\infty} e^{-h} \frac{h^{j\gamma-1}}{(j\gamma-1)!}$$

$$k = \sum_{j=1}^{\infty} e^{-h} j \left[\sum_{1=0}^{\gamma-1} \frac{h^{j\gamma+1}}{(j\gamma+1-1)!} - (1+h) \sum_{1=0}^{\gamma-1} \frac{h^{j\gamma+1}}{(j\gamma+1)!} \right].$$

The same formulas apply for y and c with proper changes in density parameters, that is, using the revised parameters for the improved parts.

Even though it is not possible to solve for ω as a function of k by eliminating h between these expressions, it is convenient to use a functional representation for such a solution. Hence, let these relationships be denoted by

$$\omega = f(k) \qquad \text{and} \qquad y = F(c).$$

Again, it is necessary to emphasize units. If the average maintenance cost per mean life is k_1 dollars, then $\omega = 1$ for the standard part. Similarly, if the average cost per mean life is c_1 dollars, then $y = 1$ for the improved part. Thus,

a unit of time is one mean life and a unit of cost is the cost of the repair of an in-service failure.

Graphs of the functions, $\omega = f(k)$ and $y = F(c)$, are shown in Figures 1 and 2 for the normal and gamma densities, respectively, for a range of density parameters. The ordinate scale is ω if the abscissa is k, while the ordinate is y if the abscissa is c. For the reasons indicated in Section 3.1.3, the ordinate is restricted to the range 0 to 1. It is obvious that the abscissa can also be restricted to the same range, since a cost ratio greater than unity means repair of an in-service failure is less expensive than a scheduled preventive maintenance action -- a very unrealistic case. However, some of the later computations will require ordinate values greater than unity.

To restrict the graphs to the unit range of the ordinate scale, the range of values greater than unity is "folded" on the unit line ω or $y = 1$ on the following basis: It will be seen later that we are always concerned with a line joining the point $(1,0)$ on the abscissa to a point on the ordinate, say $(0,m)$. The point at which this line crosses the "folded" ordinate scale is labeled with the value m. For example, a line joining points $(1,0)$ and $(.5,1)$ would pass through the point on the extended ordinate $(0,2)$, or $m = 2$. Hence, on the "folded" scale the value 2 occurs halfway between the left and right ordinate lines.
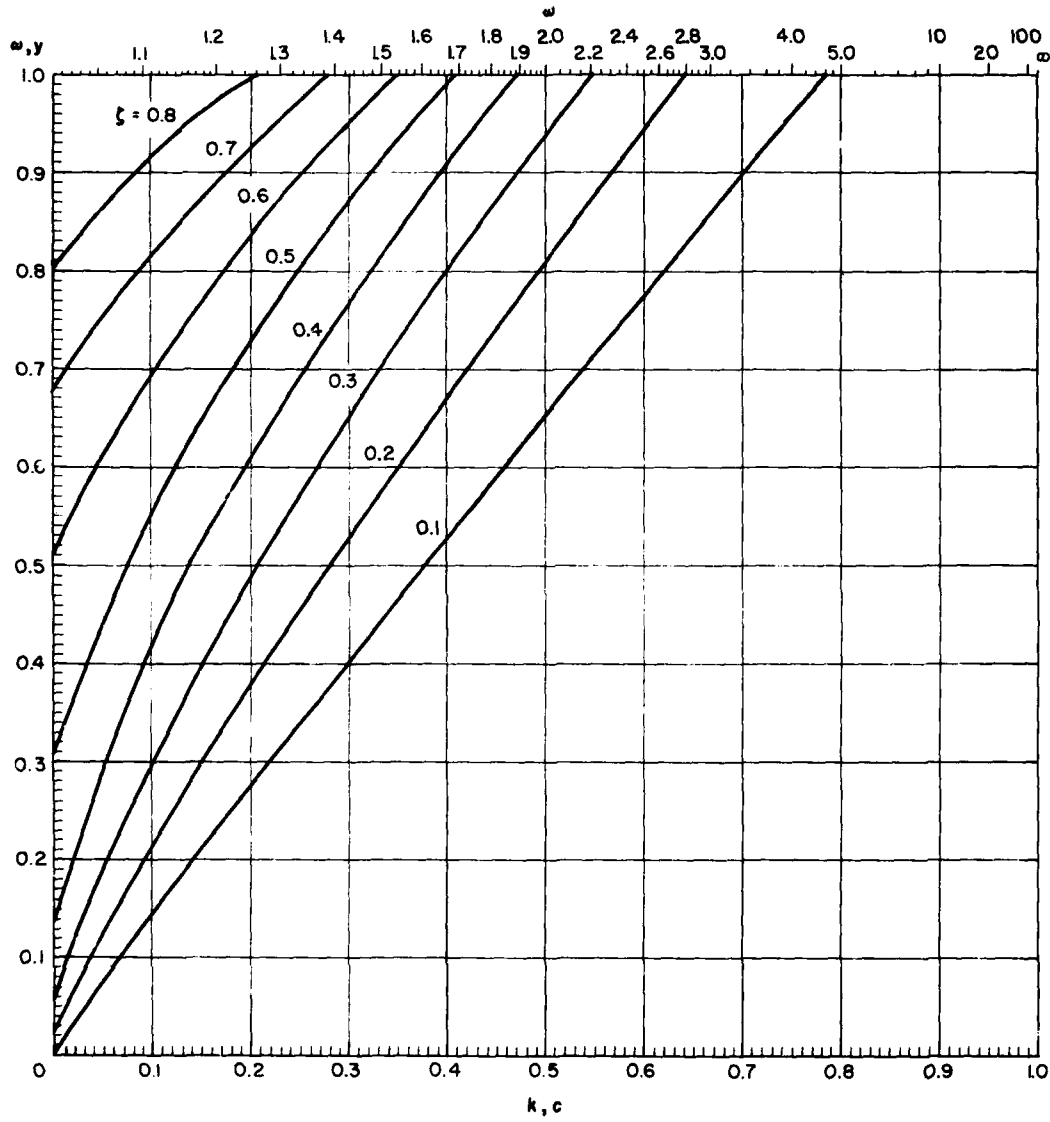
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FIGURE I

Graphs of $\omega = f(k)$ and $y = F(c)$ for selected values of ζ:
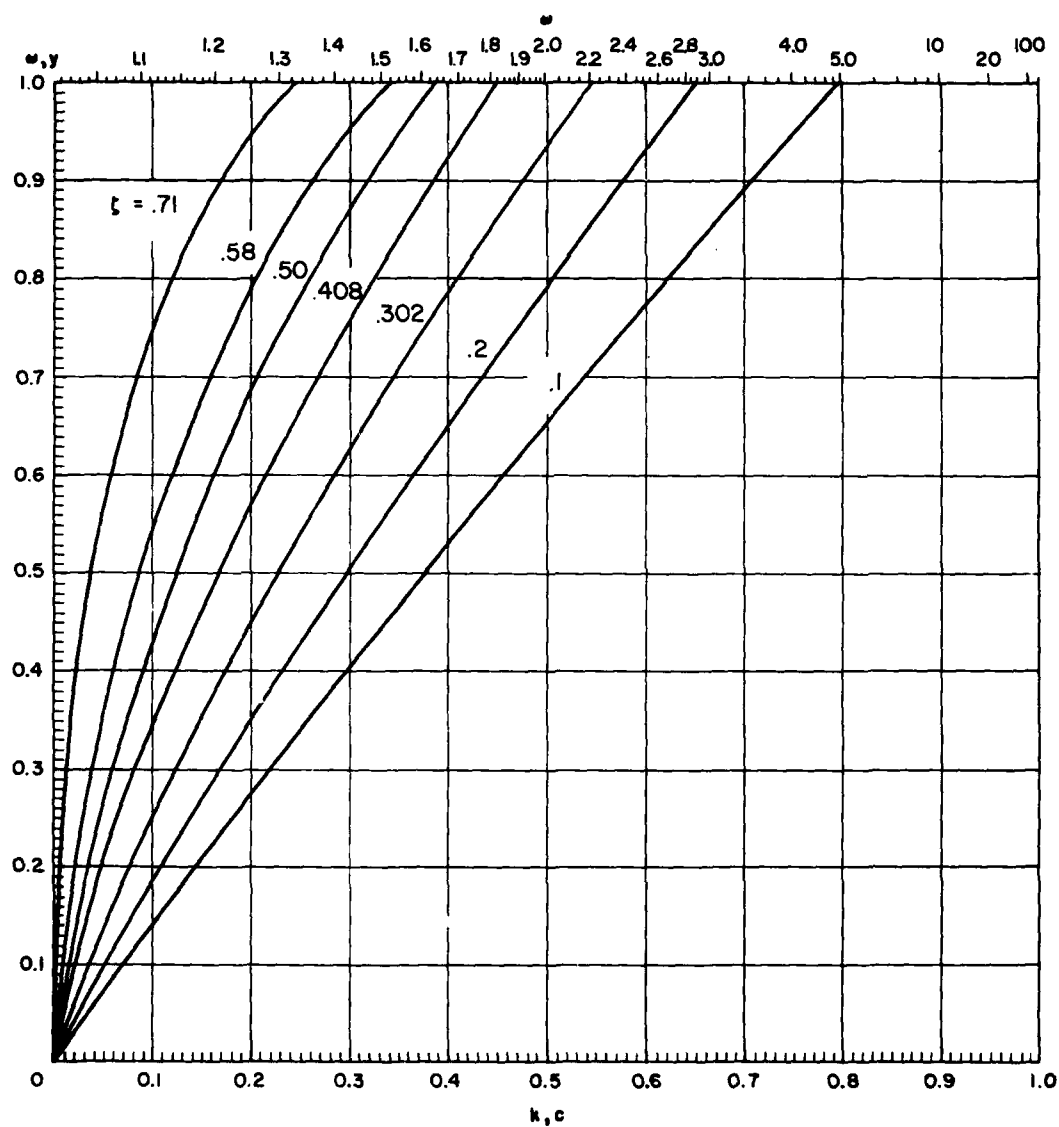Normal distribution

FIGURE 2

Graphs of $\omega = f(k)$ and $y = F(c)$ for selected values of ζ:
Gamma distribution

5. COMPUTATION OF THE COST INCREMENT

The computation of the cost increment justified by a part reliability improvement at the break-even point can now be carried out by the following sequence of steps:

(1) Obtain the basic data required. This includes determination of (a) the form of the underlying density functions for the standard and improved parts, (b) the numerical values of the parameters of these functions, (c) the ratio of standard deviations, u, and of mean lives, v, previously defined, and (d) the basic cost parameters, k_1, k_2, and the ratio $k = k_2/k_1$.

(2) Assuming that optimum preventive maintenance is being used, the value of ω can be read from the k,ω curve appropriate to the standard part as determined by its density function parameter values, and the cost ratio k.

(3) Draw the line
$$c = 1 - \frac{1-k}{v\,\omega}\, y$$
by computing the coordinates of the intercepts of the line
$$(c = 1,\ y = 0) \quad \text{and} \quad \left(c = 0,\ y = \frac{v\omega}{1-k}\right).$$

(4) Read the c,y coordinates of the point of intersection
     of the line drawn in (3) and the appropriate (c,y)
     cost curve determined by the parameter values of
     the improved-part density function.

(5) Compute the value of the cost increment, δ, from
     the formula

$$\delta = k_1\left(\frac{\omega v}{y} - 1\right) .$$

5.1 Computations by Means of a Nomograph

All of the numerical work can be accomplished by using
the nomograph, Figure 3, in connection with the k,ω and c,y
curves in Figure 1 or in Figure 2, for the normal or the gamma
density, as the case may be. The operations of multiplication,
division, and subtraction are all involved, and the nomograph is
designed to carry these out with a minimum of scale reading.
Decimal points are set for these scales just as they are in a
slide rule computation.

To illustrate the scale relationships, if a number is
located on scale c and another on scale a, the line joining the
two points will intersect scale b at the point which is the
quotient of the scale c value divided by the scale a value.
Reversing this procedure shows that a scale value on a joined
by a line through a scale value on b will intersect scale c
at a point corresponding to the product. Subtraction is per-
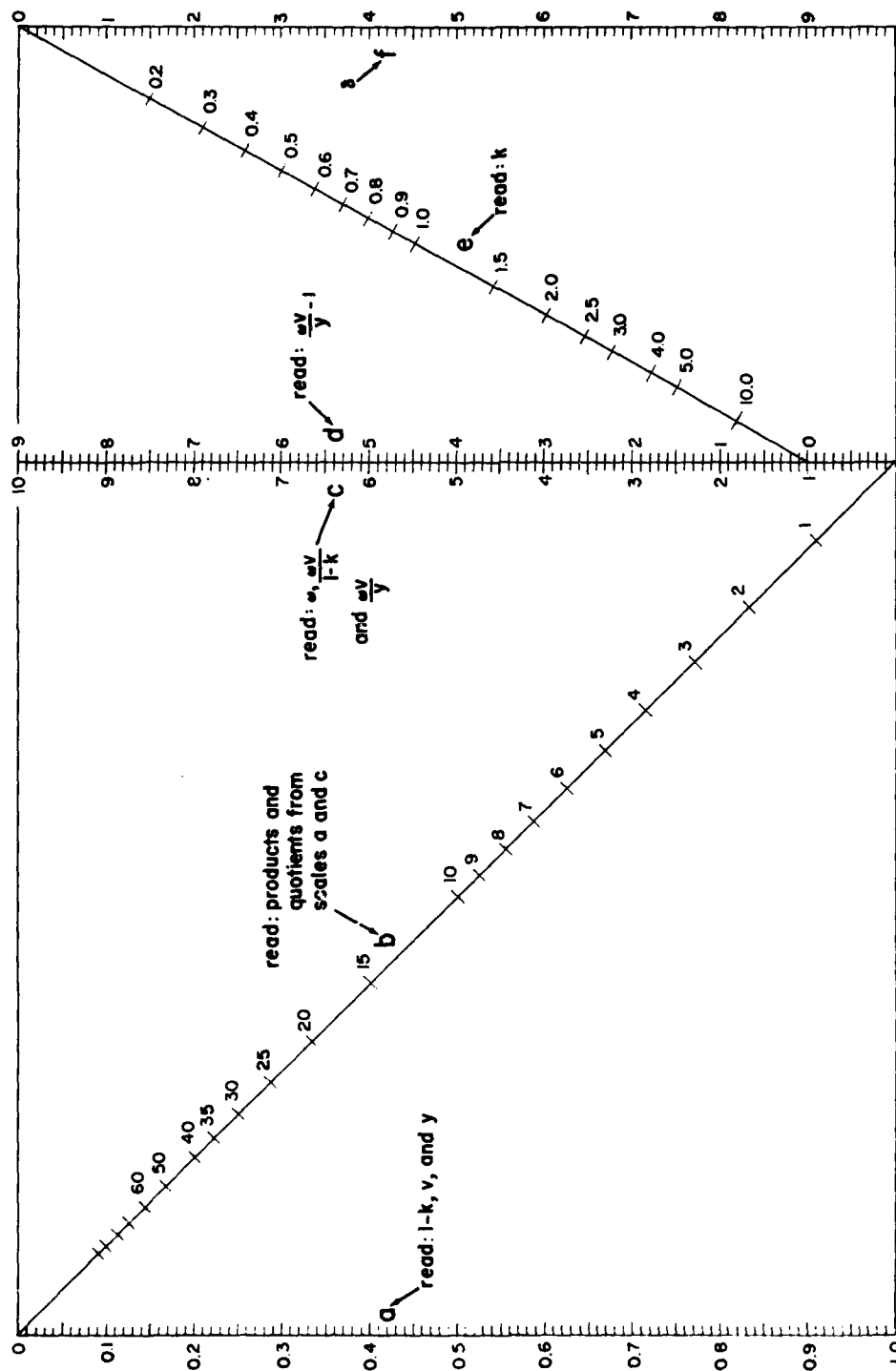formed by a renumbering technique. Thus, scale d shows the

FIGURE 3

NOMOGRAPH FOR COMPUTATION OF COST INCREMENT

values of scale c reduced by unity, a subtraction required in this particular application. Scales d, e, and f are related in the same way as scales c, b, and a, respectively, the reversal of order resulting from the sequence of steps required by the formulas in the present application.

The value of δ, the price increment at the break-even point, can be computed on the nomograph by the following steps:

(1) Connect ω on scale c to $1-k$ on scale a, giving $\frac{\omega}{1-k}$ on scale b.

(2) Connect $\frac{\omega}{1-k}$ on scale b, obtained in (1), to v on scale a, and read $\frac{\omega v}{1-k}$ on scale c.

(3) On the c,y figure, connect the points $(c = 0, y = \frac{\omega v}{1-k})$ and $(c = 1, y = 0)$ and read the coordinates of the point of intersection of this line and the appropriate c,y curve.

(4) Connect y, as determined in (3), on scale a to ω on scale c, giving $\frac{\omega}{y}$ on scale b.

(5) Connect the point $\frac{\omega}{y}$ on scale b, obtained in (4), to the value v on scale a to give $\frac{\omega v}{y}$ on scale c, which is at the value $\frac{\omega v}{y} - 1$ on scale d.

(6) Connect the point $\frac{\omega v}{y} - 1$ on scale d, found in (5) to k_1 on scale e to determine $\delta = k_1(\frac{\omega v}{y} - 1)$ on scale f.

The use of the nomograph and the cost curves can be better understood by following the steps as described above for a simple example. Let the basic parameters have the values:

$$k_1 = \$20$$
$$k = .3$$
$$\zeta_1 = .302$$
$$v = 2$$
$$u = 2.$$

Assume both the standard and improved parts have gamma time-to-failure densities.

Consider first the direct use of the formulas to compute algebraically the cost increment which can be paid for the improved part. The coefficient of variation for the time-to-failure density of the improved part is

$$\zeta_2 = \frac{u}{v} \zeta_1 = \frac{2}{2} (.302) = .302.$$

(The fact that $\zeta_1 = \zeta_2$ involves no restriction in the method and no loss in the usefulness of the example.) The value of ω corresponding to $k = .3$ is read from the gamma cost curve for $\zeta_1 = .302$ to be $\omega = .63$. It is now necessary to find the value of y for the improved part. This is accomplished by drawing a line through the points $(0, \frac{\omega v}{1-k})$ and $(1,0)$ and reading the ordinate, y, of the point where this line crosses the gamma cost curve for $\zeta_2 = .302$. Now

$$\frac{\omega v}{1-k} = \frac{(.63)(2)}{1 - .3} = 1.80,$$

and the line joins points (0, 1.80) and (1,0). It intersects the ζ_2 = .302 cost curve at the point where y = .92. The cost increment is

$$\beta = k_1\left[\frac{\omega v}{y} - 1\right] = \$20\left[\frac{(.63)(2)}{.92} - 1\right] = \$7.40.$$

Now consider the solution using the nomograph to carry out these same arithmetical operations. There is no change in the use of the gamma cost curves. First, it is necessary to read the ω value for k = .3 from the ζ_1 = .302 cost curve giving ω = .63 as before. Next join ω = .63 on the c scale to 1 - k = 1 - .3 = .7 on the a scale, reading the value $\frac{\omega}{1-k}$ = .90 on scale b. Remember that decimal point determination when using the nomograph is done in the same way as when using a slide rule. Connect this point, .90 on scale b to point v = 2 on scale a, reading $\frac{\omega v}{1-k}$ = 1.80 on the scale c. Now $\zeta_2 = \frac{u}{v}\zeta_1$ = .302 as before. Read y = .92 as the intersection of the line joining (0,1.80) to (0,1) and the gamma cost curve for ζ_2 = .302. Return to the nomograph, join ω = .63 on the c scale to y = .92 on the a scale, and read $\frac{\omega}{y}$ = .685 on the b scale. Connect point $\frac{\omega}{y}$ = .685 on the b scale to v = 2 on the a scale, giving $\frac{\omega v}{y}$ = 1.37 on the c scale, which is also $\frac{\omega v}{y}$ - 1 = .37 on the right side, the d scale. Connect this point to k_1 = \$20 on the e scale, and read \$7.40 = β on the f scale.

5.2 Computation when Type I Maintenance is Involved

The use of the nomograph was illustrated by an example in which the standard and improved parts had cost and time-to-failure density parameters which dictated Type II maintenance for both. This illustrates the computation of the first of the four cases listed below.

Case	Type of Maintenance	
	Standard Part	Improved Part
1	II	II
2	II	I
3	I	I
4	I	II

It was noted in Section 3.1.3 that the choice between the two types of maintenance depends on the value of the average cost per mean life. Thus, for the standard part, $\omega < 1$ implies Type II maintenance, while $\omega = 1$ implies Type I maintenance. The same criterion on y in place of ω determines maintenance type for the improved part. The Type I situation involves no complication in the computational procedure whether it is done by formula or by the nomograph. For the standard part, one reads $\omega < 1$ whenever the k value identifies a point on the appropriate cost curve, i.e., the curve corresponding to ζ_1. For other values of k, which are in all cases the larger values, there is no point on the curve, and one takes $\omega = 1$. This identifies Type I. The value of y for the improved part

is determined as the ordinate of the point of intersection of the ζ_2 curve and the line joining $(0, \frac{\omega v}{1-k})$ to $(1,0)$. This line crosses the curve if $y < 1$, but if it misses the curve by passing it on the right, then take $y = 1$. These two situations identify Type II and Type I maintenance, respectively. When ω and/or $y = 1$ there is a simplification of the formulas for the cost increment δ, as previously noted. The δ formulas for the four cases.are shown below.

	Case	Cost Increment
1	$(\omega<1,\ y<1)$	$\delta = k_1 \left(\frac{v\omega}{y} - 1 \right)$
2	$(\omega<1,\ y=1)$	$\delta = k_1 (v\omega - 1)$
3	$(\omega=1,\ y=1)$	$\delta = k_1 (v - 1)$
4	$(\omega=1,\ y<1)$	$\delta = k_1 \left(\frac{v}{y} - 1 \right)$

As noted previously, case 1 was illustrated by the example used in explaining the use of the nomograph. The other three cases are illustrated below, assuming that the gamma time-to-failure density applies in all instances. Note that the type of density does not affect the method, but it does identify the cost curve which is to be used.

Example for Case 2

Let the basic parameters be

$$k_1 = \$20$$
$$k = .3$$
$$\zeta_1 = .302$$
$$v = 3.33$$
$$u = 2.22.$$

Then $\zeta_2 = \frac{u}{v}\,\zeta_1 = .20$.

Now reference to the k,ω gamma cost curve for $\zeta_1 = .302$ shows that $\omega = .63$ for $k = .3$. Compute $\frac{\omega v}{1-k} = 3.00$, draw the line joining $(0,3.00)$ to $(1,0)$, and attempt to read the y coordinate of the intersection of this line with the cost curve for $\zeta_2 = .20$. In this case the line and curve do not intersect, so the improved part requires Type I maintenance, meaning $y = 1$. Then

$$\delta = k_1 \left[\omega v - 1\right] = \$20 \left[(.63)(3.33) - 1\right] = \$21.96.$$

Example for Case 3

Let

$$k_1 = \$20$$
$$k = .4$$
$$\zeta_1 = .58$$
$$v = .96$$
$$u = .50$$

For $k = .4$ and $\zeta_1 = .58$, Type I maintenance is indicated, meaning $\omega = 1$. Compute $\zeta_2 = \frac{u}{v}\,\zeta_1 = .302$, and $\frac{v}{1-k} = 1.60$. The line joining $(0,1.60)$ to $(1,0)$ intersects the

$\zeta_2 = .302$ curve at a point with y coordinate of .865, indicating Type II maintenance for the improved part. The dollar value of the reliability increment is

$$\delta = \$20 \left[\frac{.96}{.865} - 1 \right] = \$2.20.$$

It is interesting to note that the mean life, or mean time to failure, of the improved part is actually less than that of the standard part, as indicated by the fact that v is less than unity. The reliability improvement results from the important decrease in the standard deviation, which, in turn, implies an improvement in the accuracy of failure prediction. In other words, more accurate prediction of failures makes possible a decrease in maintenance cost through part replacement on a scheduled basis prior to in-service failure.

Example for Case 4

For thi. ample

$$k_1 = \$20$$
$$k = .4$$
$$\zeta_1 = .58$$
$$v = 1.2$$
$$u = .844.$$

The $\zeta_1 = .58$ cost curve shows that $\omega = 1$ for $k = .4$, indicating Type I maintenance. Now

$$\zeta_2 = \frac{u}{v} \zeta_1 = .408, \text{ and } \frac{\omega v}{1-k} = 3.$$

In this case the line joining (3,0) to (0,1) does not
intersect the cost curve for ζ_2 = .408, and hence y = 1,
indicating Type I maintenance for the improved part.
Then the cost increment is

$$\delta = \$20 \ (1.2 - 1) = \$4.00.$$

5.3 The Break Point Between Type II and Type I Maintenance; The Transition from Case 1 to Case 2

The four cases were automatically identified in the com-
putational procedure as indicated above. It is of some theo-
retical interest, however, to consider the range of k values
for the standard part and to break this range into two inter-
vals corresponding to Type II and Type I maintenance for the
improved part. This can be done in the manner described below.

From the basic parameters ζ_1, v, and u, compute $\zeta_2 = \frac{u}{v} \zeta_1$.
Consider the problem of relating the k range on the ζ_1 curve
to the break point between Type II and Type I maintenance for
this value of ζ_2. Recall that the computational procedure
assumed that k was given and that the point on the ζ_2 curve was
to be computed. For present purposes the procedure must be
reversed. Actually the point on the ζ_2 curve is known: it is
the point where the curve intersects the line y = 1, for that
is the division between Type II maintenance (y < 1) and Type I
maintenance (y = 1). Denote this point by (c,1). Previously
it was observed that the (k,ω) and (c,y) values satisfy

$$c = 1 - \frac{1-k}{v \ \omega} y.$$

In the usual computation, with (k,ω) known, this equation was used to determine (c,y). Now the procedure will be to let $c = c'$ and $y = 1$, and to compute k and ω. Thus

$$c' = 1 - \frac{1-k}{v\,\omega} \text{, or } v\omega(1-c') = 1-k.$$

This is a line in variables k and ω which can be plotted as the line through the two points

$$\left(k = 0, \ \omega = \frac{1}{v(1-c')} \right) \text{ and } (k = 1, \ \omega = 0).$$

Since k and ω must also be the coordinates of a point on the ζ_1 cost curve, the graphical solution is obtained by finding the coordinates of the point of intersection of the line and the ζ_1 cost curve.

The method can be easily understood from a numerical illustration. Let $\zeta_1 = .71$, $v = 2.35$, and $u = 1$. Then $\zeta_2 = \frac{u}{v}$ and $\zeta_1 = .302$. The $\zeta_2 = .302$ cost curve passes through the point $c' = .545$, $y = 1$. Hence the line, $\omega v(1-c') = 1 - k$, is

$$2.35\omega(1 - .545) = 1 - k, \text{ or } k = 1 - 1.07\omega.$$

This is the line through points

$$(k = 0, \ \omega = .934) \text{ and } (k = 1, \ \omega = 0)$$

and it intersects the ζ_1 cost curve in the point $(k = .126, \ \omega = .815)$.

Now if $k < .126$, the improved part will require Type II maintenance, while if $k \geq .126$ it will require Type I maintenance. A numerical check will establish that $k = .126$ does

generate a line through the terminal point on the ζ_2 cost curve on the line y = 1.

It is of interest that the "folded" ordinate scale can also be used to determine the k value intervals. In the numerical example, the ζ_2 = .302 curve terminates at the value 2.2 on the folded scale. This means that the line in question passes through the point (0, 2.2) or, in other words, $\frac{v\omega}{1-k}$ = 2.2. Since v = 2.35, then $\frac{2.35}{1-k}$ = 2.2. This reduces to k = 1 - 1.07ω, the same relationship obtained previously. The balance of the solution duplicates that given above.

5.4 The Cost Increment for an Exponential Time-to-Failure Density

The frequent assumption of an exponential time-to-failure density justifies special consideration of this case. Since the exponential constant hazard rate means that failure probability is independent of age, there is never any need for preventive-maintenance part replacement. This means that Type I maintenance is always applicable, and the expected cost is k_1 per mean life. Therefore, if an improved part becomes available, the dollar value is vk_1 where v is the ratio of mean life as defined previously. Assume for example, that a part costs $20 and that its mean life is 100 hours. One could afford to pay $25 for a part with a mean life of 125 hours since the ratio of these costs is equal to the ratio of the mean lives. One need not consider variance in this case, since, in a one-parameter system such as the exponential, the variance is functionally expressible in terms of the mean.

Previous cost studies have frequently treated reliability
cost trade-offs in terms of these ratios of mean time to fail-
ure. Such analyses provide only crude estimates in those
instances in which a well-chosen preventive maintenance policy
makes use of knowledge of deterioration characteristics.
However, if Type I maintenance is used even though Type II
maintenance would be less costly, then the ratio of mean lives
is properly equated to the ratio of costs. *

5.5 The Sensitivity of the Cost Increment for Improved Reliability to the Underlying Density

The usefulness of the method for computing the dollar
value of improved reliability would be greatly enhanced if it
were not sensitive to the exact form of the underlying density,
but, instead, were primarily influenced by a relatively small
number of density parameters. It would be natural to suspect,
for example, that the coefficient of variation might be the
primary determinant of the cost increment. It is obvious,
however, that such a parameter is not the sole determinant, and
therefore it is suggested that the simplest check on the sen-
sitivity is merely a set of computations of cost increments
for a selection of parameter values in relation to a number of
underlying density types. Results of these computations are
shown in Figures 4, 5, and 6.

It has been necessary to restrict this analysis to only
two time-to-failure density types, the normal and the gamma.
The reason for this restriction, as previously explained, is

FIGURE 4

IMPROVED-PART COST INCREMENT (δ) AS A FUNCTION OF COST RATIO (k)
FOR NORMAL AND GAMMA TIME-TO-FAILURE DENSITIES:
COEFFICIENT OF VARIATION (ζ_2) = 0.1
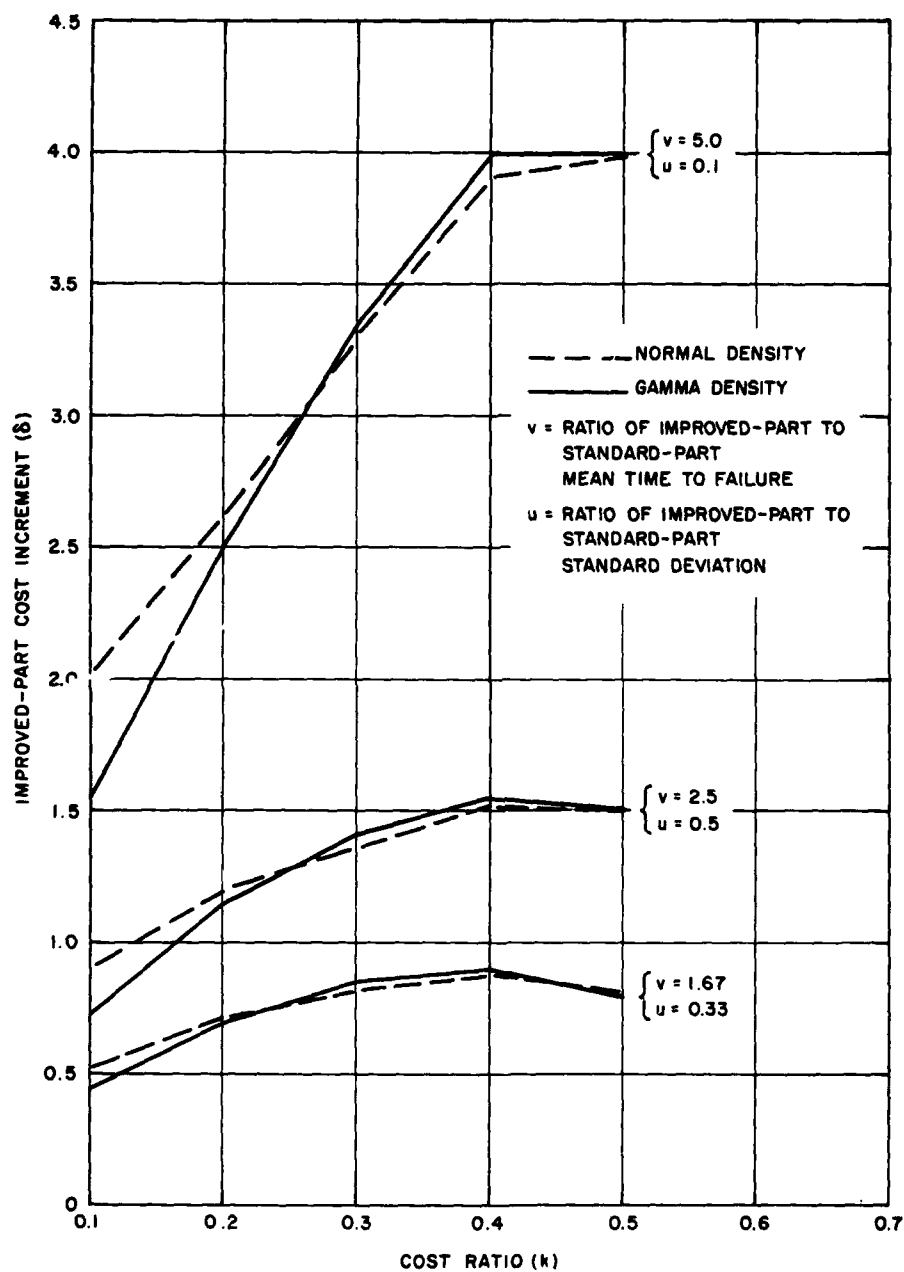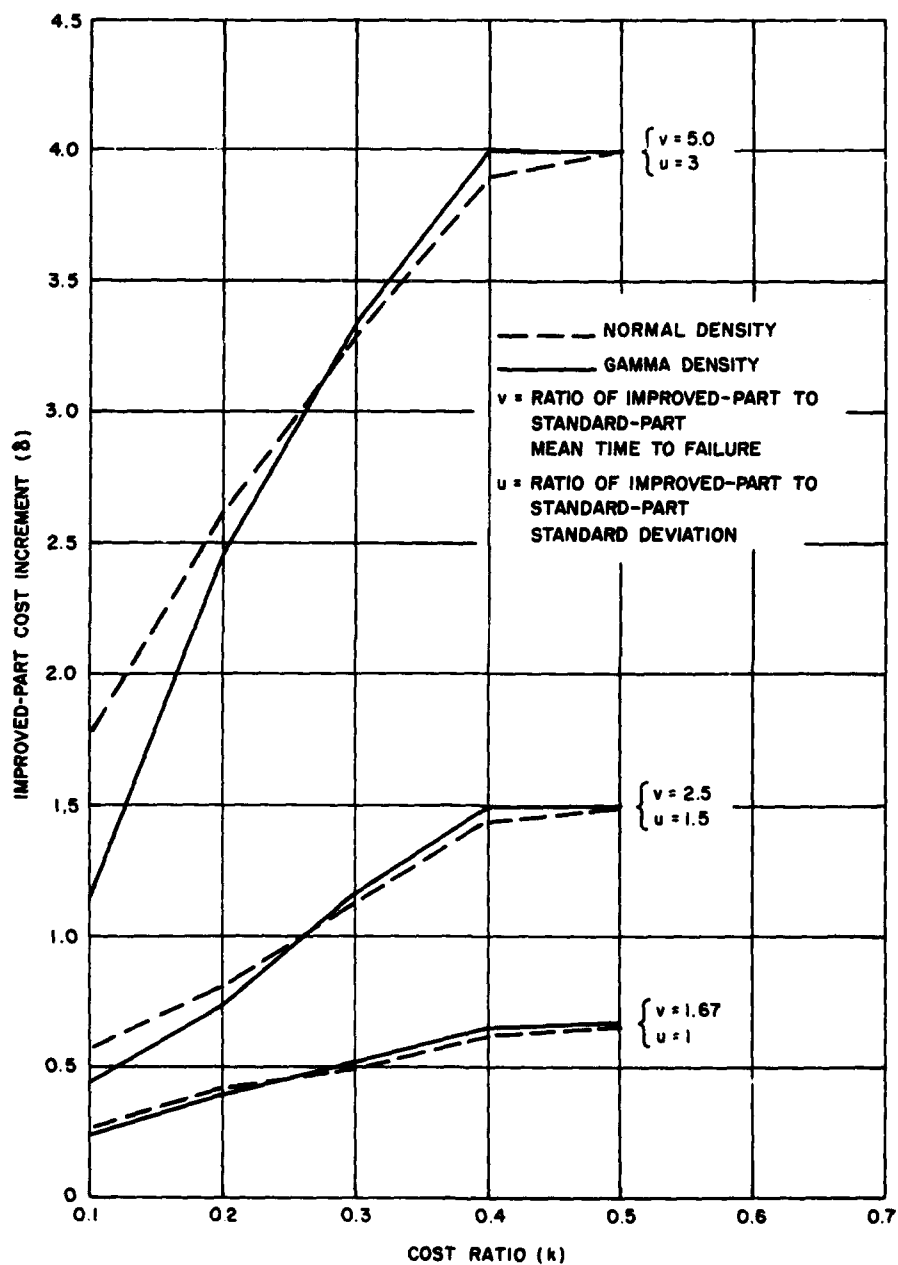
FIGURE 6

IMPROVED-PART COST INCREMENT (δ) AS A FUNCTION OF COST RATIO (k)
FOR NORMAL AND GAMMA TIME-TO-FAILURE DENSITIES:
COEFFICIENT OF VARIATION (ζ_2) = 0.3

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that Monograph 7 developed formulas only for these two
densities. It is hoped that later studies can include others.
While Monograph 7 did contain a special discussion of the
exponential density, the exponential is of trivial significance
with respect to the present problem, since the cost increment
is directly proportional to the increase in mean life.

The examples chosen here exclude large values of the
coefficient of variation because of the peculiar properties
of the normal distribution in relation to "negative" times-to-
failure. For a relatively complete discussion of this problem,
the reader is referred to Monograph 7. Briefly, the explana-
tion is that the time range from $-\infty$ to $+\infty$, which has been
assumed for the normal density, is somewhat unrealistic from
an engineering viewpoint. It is not easy to explain "negative"
times when in fact they represent installation or zero-time
failures. This means that a correct analysis would make use
of a truncated density of the form

$$u(t) = \frac{1}{\sigma \sqrt{2\pi}} \, e^{-\frac{(t-\xi)^2}{2\sigma^2}} \qquad \text{for } 0 < t, \text{ and}$$

$$u(t) = \int_{-\infty}^{0} \frac{1}{\sigma \sqrt{2\pi}} \, e^{-\frac{(t-\xi)^2}{2\sigma^2}} \, dt \qquad \text{for } t = 0$$

in place of the form actually used,

$$u(t) = \frac{1}{\sigma \sqrt{2\pi}} \, e^{-\frac{(t-\xi)^2}{2\sigma^2}} \qquad -\infty < t < \infty.$$

The obvious reason for not using the truncated form is the mathematical complexity introduced by the integral of the normal density over the range with one finite limit. The error involved is small if the integral itself is small; and the latter situation obtains if the coefficient of variation, $\zeta = \frac{\sigma}{\xi}$, is small. This restriction merely says that the probability that t is in the range of positive values is very nearly unity in case ζ is small.

Cost increment computations were made for the combinations of parameters shown below, for both the normal and gamma densities. It was assumed for this purpose that ζ_1 had the value 0.5. The field of the tabulation gives the ratio of standard deviation (u) employed with each ratio of mean

	ζ_2		
v	.1	.2	.3
1.67	.33	.67	1.0
2.50	.50	1.00	1.5
5.00	1.00	2.00	3.0

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time to failure (v) and each coefficient of variation (ζ_2).
The cost increment was computed for each of the above com-
binations for five values of the cost ratio k:0.1, 0.2, 0.3,
0.4, and 0.5. The value of k_1 was assumed to be $1.00, which
is equivalent to computing the quantity $x = \delta/k_1$, discussed
in Section 3.1.2. It was found that this range of values of k
included examples of both Type I and Type II maintenance.

Figures 4, 5, and 6 show the values of the cost increment
plotted against the cost ratio. There is a surprising lack of
sensitivity to the underlying density. The only cases of any
sizeable difference are found in the curves for which $v = 5.0$
and the cost ratio is $k = 0.1$. This suggests that the cost
increment for improved reliability is determined primarily by
the coefficient of variation and the cost parameters, but is
very insensitive to the exact form of the underlying density
function. It is recognized that these few examples are not
conclusive, and that further study is definitely needed. The
examples are sufficiently striking, however, to lead to the
belief that exact determination of the form of the underlying
density is not required if one is given the choice between a
standard part and an improved part which is available for a
cost increment less than the break-even value computed from
the best-fitting gamma or normal density. This conclusion is
based partly on the knowledge that these two densities are
sufficiently flexible to provide reasonably good fits to a
large percentage of the observed time-to-failure data.

This illustration of lack of sensitivity to the underlying density is important as an indication of a much larger area for research. Underlying time densities are basic in the derivation of numerous indices and measures of the many factors associated with the general concept of system effectiveness. In many instances, it has been assumed that the exact form of the density is critical. The analysis in this monograph suggests that the assumption may be questionable in many instances, and that the exact density may not be overly critical in such procedures as prediction, preventive maintenance scheduling, and average cost estimation.

6. SUMMARY AND CONCLUSIONS

On the basis of (1) a criterion for determining a preventive maintenance schedule and (2) the associated curves relating average cost per mean life to the fundamental cost parameter, this study develops a nomograph permitting computation of the maximum cost increment which can be paid for improved reliability. The nomograph is distribution-free. The basic cost curves, however, depend on the preventive maintenance policy and the underlying density. Some examples suggest very strongly that the cost increment is sensitive to the coefficient of variation but that it is relatively insensitive to the exact form of the density function.

NOTATION FOR COST FORMULAS

Item	Standard Part	Improved Part
Mean time to failure	m	mv
Standard deviation	σ	σu
Coefficient of variation	ζ_1	ζ_2
Time between preventive maintenance actions	h_1	h_2
Cost parameters:		
Repair of an in-service failure	k_1	c_1
Cost of a preventive maintenance part replacement	k_2	c_2
Cost ratio	$k = \dfrac{k_2}{k_1}$	$c = \dfrac{c_2}{c_1}$
Improved-part cost increment		δ
Expected average cost per mean life (optimum maintenance)	$k_1 \omega$	$c_1 y$